A 3-regions new economic geography model in discrete time

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Abstract — **In this paper we present a 3-regions new economic geography model. We assume that the migration decisions of the mobile productive factor occur in between time intervals. The corresponding dynamic process is summarized by a two-dimensional dynamical map. We study this map identifying fours different types of fixed points, their local stability properties and their basins of attraction. We also present numerical simulations showing the existence of periodic and chaotic attractors.**

I. Introduction

The footloose entrepreneurs (FE) model proposed by Forslid and Ottaviano (2003) is a variant of the well-known Core-Periphery (CP) model of Economic Geography proposed by the 2008 Nobel Prize Paul Krugman (see Krugman, 1991). The FE model maintains the basic structure of the CP model. The economy is composed of two symmetric/identical regions and two productive sectors: agriculture and manufacturing. The first is perfectly competitive, whereas in the second increasing returns prevails. Moreover, distance, in the form of transport costs, plays a crucial role in determining the price difference between locally produced and imported manufactured goods. Finally, both assume the existence of a mobile factor of production (labour in the CP model and entrepreneurs/human capital/skilled labour in the FE model) whose decisions to migrate affect both the location of the manufacturing sector and the size of the market. The main difference is that in the CP model the mobile factor enters in production both as a fixed and as a variable cost component; instead, in the FE model it enters only as a fixed cost component. This assumption reduces substantially the analytical complexity of the FE model. Both CP and FE models were originally framed in continuous time. More recently Currie and Kubin (2006) and Commendatore, Currie and Kubin (2008) presented discrete time versions of the CP and FE models, showing that while preserving many of the most interesting features of their continuous time counterparts (hysteresis, multiple equilibria, catastrophic changes) they enjoyed additional features simply due to the different time framework (chaotic dynamics, multiple attractors of any periodicity, agglomeration via volatility). The objective of this paper is to extend the discrete time version of the symmetric FE model presented in Commendatore, Currie and Kubin (2008) to the case of three regions. As stated recently by Fujita and Thisse (2009) the existence of more than two regions may involve effects that cannot emerge in a two regions context. It seems natural therefore to verify the emergence of these effects and of further dynamic effects in the discrete time version of the FE model when three regions are involved. This paper, that is concerned with the case of three symmetric regions – that is, the regions are identical except for the distribution of the manufacturing activity –, represents a step in this direction.

II. Basic Framework

The economic system is composed of three regions ($r = 1, 2, 3$). In each region an agricultural sector (A) and, potentially, a manufacturing sector (M) are localised. Production involves the use of two factors of production. Unskilled labour (L), that does not migrate, is equally distributed among the regions. Thus, $L/3$ unskilled workers reside in each region, where L is the total number of unskilled workers in the overall economy. Entrepreneurs (N), instead, are mobile across regions.

The three regions are also characterised by the same tastes, technology and transport costs. The representative consumer's utility function is:

$$U = C_A^{1-\mu} C_M^{\mu} \tag{1}$$

where C_A and C_M correspond to the consumption of the homogeneous agricultural good and of a composite of manufactured goods:

$$C_M = \sum_{i=1}^{n} d_i^{\frac{\sigma-1}{\sigma}} \tag{2}$$

where d_i is the consumption of good i, n is the total number of manufactured goods and $\sigma > 1$ is the constant elasticity of substitution; the lower σ, the greater the consumers' taste for variety. The exponents in the utility function $1-\mu$ and μ indicate, respectively, the invariant shares of disposable income devoted to the agricultural and manufactured goods, with $0 < \mu < 1$.

The manufacturing sector involves Dixit-Stiglitz monopolistic competition. In our context, each firm requires a fixed input of an entrepreneur to operate and β units of unskilled labour for each unit produced. Since one entrepreneur is needed for each firm, the total number of firms always equals the total number of entrepreneurs. Moreover, because of consumers' preference for variety and increasing returns in production, a firm would always produce a variety different from those produced by others. It follows that the number of varieties always equals the number of firms. Denoting the share of entrepreneurs located in region r in period t by $\lambda_{r,t}$ and by N the total number of entrepreneurs, the number of regional varieties produced in period t in region r is

$$n_{r,t} = \lambda_{r,t} N \qquad (3)$$

where $r = 1, 2, 3$, $0 \le \lambda_{r,t} \le 1$ and $\sum_{r=1}^{3} \lambda_{r,t} = 1$.

Transportation of the agricultural product between regions is costless. Transport costs for manufactures take an iceberg form: if one unit is shipped between regions r and s $1/T_{rs}$ arrives, where $T_{rs} \ge 1$, $r, s = 1, 2, 3$ and $r \ne s$. With identical trade costs among the regions, we have that

$$T_{rs} = T \ge 1 \quad for \quad r \ne s \quad T_{rr} = 1 \quad otherwise.$$

Finally, in order to simplify the notation, we introduce the "trade freeness" parameter, defined as $\phi_{rs} \equiv T_{rs}^{1-\sigma}$.

III. SHORT-RUN GENERAL EQUILIBRIUM

The short-run equilibrium in period t is characterized by a given spatial allocation of entrepreneurs across the regions, $\lambda_{r,t}$. In a short-run general equilibrium, which is established instantaneously in each period, supply equals demand for the agricultural commodity and each manufacturer meets the demand for its variety. As a result of Walras's law, simultaneous equilibrium in the product markets implies equilibrium in the regional labour markets.

With zero transport costs, the agricultural price is the same across regions. Denoting by Y the income of the overall economy, that (as confirmed below) is invariant over time, total expenditure on the agricultural product is $(1-\mu)Y$. Assuming $(1-\mu)Y > 2L/3$ all regions produce the agricultural commodity. Since competition results in zero agricultural profits, the short-run equilibrium nominal wage in period t is equal to the agricultural product price and therefore is always the same across regions. Setting this wage/agricultural price equal to 1, it becomes the numeraire in terms of which the other prices are defined. Facing a wage of 1, each manufacturer has a marginal cost of β. Each maximizes profit on the basis of a perceived price elasticity of $-\sigma$ and sets a local (mill) price p for its variety, given by

$$p = \frac{\sigma}{\sigma - 1}\beta \qquad (4)$$

The effective price paid by consumers in region r for a variety produced in region s is pT_{rs}. The regional manufacturing price index facing consumers in region r is given by

$$P_{r,t} = \left(\sum_{s=1}^{3} n_s p^{1-\sigma} T_{rs}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

With identical trade costs across regions, we can write

$$P_{r,t} = \Delta_{r,t}^{\frac{1}{1-\sigma}} N^{\frac{1}{1-\sigma}} p \qquad (5)$$

where $\Delta_{r,t} = \lambda_{r,t} + \phi(1-\lambda_{r,t})$ and $\phi \equiv T^{1-\sigma}$.

The demand per variety in region r is

$$d_{r,t} = \left(\sum_{s=1}^{3} \mu Y_{s,t} P_{s,t}^{\sigma-1} T_{r,s}^{1-\sigma} \right) p^{-\sigma} = \left(\sum_{s=1}^{3} s_{s,t} P_{s,t}^{\sigma-1} T_{r,s}^{1-\sigma} \right) \mu Y p^{-\sigma} \qquad (6)$$

where $Y_{s,t}$ represents income and expenditure in region s in period t, $s_{s,t} \equiv Y_{s,t}/Y$ denotes region s' share in expenditure in period t and $s = 1, 2, 3$.

Short-run general equilibrium in region r requires that each firm meets the demand for its variety. For a variety produced in region r,

$$x_{r,t} = d_{r,t} \qquad (7)$$

where $x_{r,t}$ is the output of each firm located in region r. From equation (4), the short-run equilibrium operating profit per variety/entrepreneur in region r is

$$\pi_{r,t} = p x_{r,t} - \beta x_{r,t} = \frac{p x_{r,t}}{\sigma} \qquad (8)$$

Since profit equals the value of sales times $1/\sigma$ and since total expenditure on manufacturers is μY, the total profit received by entrepreneurs is $\mu Y/\sigma$. Total income is $Y = L + \mu Y/\sigma$, so that

$$Y = \frac{\sigma L}{\sigma - \mu} \qquad (9)$$

Total profit is therefore $\mu L/(\sigma - \mu)$. Equation (9) confirms that total income is invariant over time. From (9), $(1-\mu)Y > 2L/3$ is equivalent to $2\mu + \sigma - 3\mu\sigma > 0$. The latter is the (sufficient) non-full-specialization condition expressed in terms of the utility parameters.

Using (4) to (9), the short-run equilibrium profit in region r is determined by the spatial distribution of entrepreneurs and the regional expenditure shares:

$$\pi_{r,t} = \left(\sum_{s=1}^{3} \mu Y_{s,t} P_{s,t}^{\sigma-1} T_{rs}^{1-\sigma} \right) \frac{p^{1-\sigma}}{\sigma} = \left(\sum_{s=1}^{3} s_{s,t} P_{s,t}^{\sigma-1} \phi_{rs} \right) p^{1-\sigma} \frac{\mu Y}{\sigma}$$

With identical trade costs across regions, we can write

$$\pi_{1,t} = \frac{\mu}{\sigma} \frac{Y}{N} \left(\frac{s_1}{\Delta_{1,t}} + \phi \frac{s_2}{\Delta_{2,t}} + \phi \frac{s_3}{\Delta_{3,t}} \right) \qquad (10)$$

$$\pi_{2,t} = \frac{\mu}{\sigma} \frac{Y}{N} \left(\phi \frac{s_1}{\Delta_{1,t}} + \frac{s_2}{\Delta_{2,t}} + \phi \frac{s_3}{\Delta_{3,t}} \right) \qquad (11)$$

$$\pi_{3,t} = \frac{\mu}{\sigma} \frac{Y}{N} \left(\phi \frac{s_1}{\Delta_{1,t}} + \phi \frac{s_2}{\Delta_{2,t}} + \frac{s_3}{\Delta_{3,t}} \right) \qquad (12)$$

Regional incomes/expenditures are

$$Y_{r,t} = \frac{L}{3} + \lambda_{r,t} N \pi_{r,t} \qquad (13)$$

Using (4) to (13) and taking into account that $\lambda_{3,t} = 1 - \lambda_{1,t} - \lambda_{2,t}$, region r's share in total expenditure $s_{r,t}$ can be expressed in terms of $\lambda_{1,t}$ and $\lambda_{2,t}$:

$$s_{1,t} = \frac{\dfrac{\sigma-\mu}{3} + \lambda_{1,t}\mu\phi\left(\dfrac{s_{2,t}}{\Delta_{2,t}} + \dfrac{1-s_{2,t}}{\Delta_{3,t}} \right)}{\sigma - \lambda_{1,t}\mu\left(\dfrac{1}{\Delta_{1,t}} - \dfrac{\phi}{\Delta_{3,t}} \right)} \qquad (14)$$

$$s_{2,t} = \frac{\dfrac{\sigma-\mu}{3} + \lambda_{2,t}\mu\phi\left(\dfrac{s_{1,t}}{\Delta_{1,t}} + \dfrac{1-s_{1,t}}{\Delta_{3,t}}\right)}{\sigma - \lambda_{2,t}\mu\left(\dfrac{1}{\Delta_{2,t}} - \dfrac{\phi}{\Delta_{3,t}}\right)} \tag{15}$$

$$s_{3,t} = 1 - s_{1,t} - s_{2,t} \tag{16}$$

Given that the agricultural price is 1, the real income of an entrepreneur in region r is:

$$\omega_{r,t} = \pi_{r,t} P_{r,t}^{-\mu} \tag{17}$$

IV. Entrepreneurial Migration and the Complete Dynamical Model

Taking into account the constraint $0 \le \lambda_{r,t+1} \le 1$, the complete dynamical system is summarized by the following piecewise smooth two-dimensional map

$$\lambda_{1,t+1} = Z(\lambda_{1,t}, \lambda_{2,t}) = \begin{cases} 0 & if \quad F(\lambda_{1,t}, \lambda_{2,t}) < 0 \\ F(\lambda_{1,t}, \lambda_{2,t}) & if \quad 0 \le F(\lambda_{1,t}, \lambda_{2,t}) \le 1 \\ 1 & if \quad F(\lambda_{1,t}, \lambda_{2,t}) > 1 \end{cases} \tag{18}$$

$$\lambda_{2,t+1} = Y(\lambda_{1,t}, \lambda_{2,t}) = \begin{cases} 0 & if \quad H(\lambda_{1,t}, \lambda_{2,t}) < 0 \\ H(\lambda_{1,t}, \lambda_{2,t}) & if \quad 0 \le H(\lambda_{1,t}, \lambda_{2,t}) \le 1 \\ 1 & if \quad H(\lambda_{1,t}, \lambda_{2,t}) > 1 \end{cases} \tag{19}$$

where the explicit form of the central dynamic equations is obtained by mimicking the replicator dynamics, widely used in evolutionary game theory:

$$F(\lambda_{1,t}, \lambda_{2,t}) = \lambda_{1,t}\left(1 + \gamma\frac{\omega_{1,t} - \sum_{s=1}^{3}\lambda_{s,t}\omega_{s,t}}{\sum_{s=1}^{3}\lambda_{s,t}\omega_{s,t}}\right)$$

and (20)

$$H(\lambda_{1,t}, \lambda_{2,t},) = \lambda_{2,t}\left(1 + \gamma\frac{\omega_{2,t} - \sum_{s=1}^{3}\lambda_{s,t}\omega_{s,t}}{\sum_{s=1}^{3}\lambda_{s,t}\omega_{s,t}}\right)$$

and where $\lambda_{3,t+1}$ is residual to (complement to 1 with respect to) $\lambda_{1,t+1}$ and $\lambda_{2,t+1}$. More precisely:

$$\lambda_{3,t+1} = 1 - \lambda_{1,t+1} - \lambda_{2,t+1} =$$
$$= \begin{cases} 0 & if \quad 1 - F(\bullet,\bullet) - H(\bullet,\bullet) < 0 \\ 1 - F(\bullet,\bullet) - H(\bullet,\bullet) & if \quad 0 \le 1 - F(\bullet,\bullet) - H(\bullet,\bullet) \le 1 \\ 1 & if \quad 1 - F(\bullet,\bullet) - H(\bullet,\bullet) > 1 \end{cases} \tag{21}$$

According to the above equations, the migration of entrepreneurs at the transition between period t and period t+1 depends on a comparison between the real income gained in a region and the weighted average of the incomes in all regions.

V. Fixed Points – Existence, Local Stability and Basins of Attraction

In addition to the boundary fixed points, that are obvious from the specifications in equations (18) and (19), numerical explorations suggest that three other types of fixed points exist: 3-regions symmetric fixed points, 3-regions asymmetric fixed points and 2-regions symmetric fixed points. In the following figure 1, which depicts equations (18) and (19) (after inserting equation (20)) for $\lambda_{i,t+1} = \lambda_{i,t}$ and for $\phi = 0.275$, $\mu = 0.45$ and $\sigma = 2.5$, intersections of the lines determine fixed points of the dynamic system.

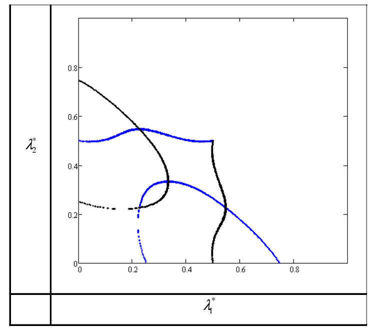

Fig. 1. Fixed points of the dynamic system

1. Boundary fixed points

In the long-run all industrial activity is agglomerated only in one region, determining a so-called Core-Periphery (CP) or boundary equilibrium:

(1, 0, 0), (0, 1, 0), (0, 0, 1)

The existence of this type of fixed points can be easily verified from equations (18) and (19) by substitution; the Jacobian evaluated at (1,0,0) is given by

$$J^B = \begin{pmatrix} j_{11} & 0 \\ 0 & j_{22} \end{pmatrix} \quad \text{with} \quad j_{11} = j_{22} \tag{22}$$

Therefore, the two eigenvalues are identical and given by

$$j_{11} = j_{22} = 1 - \gamma\left(1 - \phi^{\frac{\mu}{\sigma-1}-1}\frac{(\sigma-\mu)(1+\phi) + (2\mu+\sigma)\phi^2}{3\sigma}\right) \tag{23}$$

This Eigenvalue is between –1 and +1 for

$$\frac{\gamma}{\gamma-2}\frac{3\sigma}{(\sigma-\mu)(\phi+1)+(2\mu+\sigma)\phi^2} < \phi^{\frac{\mu-\sigma+1}{\sigma-1}} < \frac{3\sigma}{(\sigma-\mu)(\phi+1)+(2\mu+\sigma)\phi^2} \tag{24}$$

The following properties hold:

a) for sufficiently low values of γ the left hand inequality in expression (24) is satisfied and the Eigenvalue is greater than –1. More specifically, this inequality is satisfied for

$$\gamma < \gamma^{CP,-1} = \frac{6\sigma}{3\sigma - \phi^{\frac{\mu-\sigma+1}{\sigma-1}}\left[\phi^2(2\mu+\sigma) - (\sigma-\mu)(1+\phi)\right]} \tag{25}$$

b) for $1 < \sigma < 1 + \mu < 2$ the right hand inequality in expression (24) is always satisfied and the Eigenvalue is less than +1;

c) for $1 < 1 + \mu < \sigma$ it can be shown that the right hand inequality in expression (24) is satisfied for sufficiently high values of ϕ and violated for low values. Therefore, in this case the Eigenvalue is less than +1 only for sufficiently high values of ϕ. It is not possible to explicitly specify the corresponding bifurcation value for trade freeness, $\phi^{CP,+1}$.

2. 3-regions symmetric interior fixed points

In the long-run equilibrium industrial activity is equally split among the three regions:

$$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) \tag{26}$$

The existence of this fixed point can be easily verified and its (local) stability can be determined by the Jacobian evaluated at this fixed point, which is given by

$$J=\begin{pmatrix} j_{11} & 0 \\ 0 & j_{22}\end{pmatrix} \quad \text{with} \quad j_{11}=j_{22} \tag{27}$$

Therefore, the two eigenvalues are identical and given by:

$$j_{11}=j_{22}$$
$$=1-\gamma\frac{(1-\phi)\left(\mu-\sigma+\left(\mu^2+\sigma^2\right)(1-\phi)-2\mu\sigma+2\mu\phi+\sigma\phi-4\mu\sigma\phi\right)}{(\sigma-1)(2\phi+1)(\sigma-\mu+\mu\phi+2\sigma\phi)} \tag{28}$$

The following properties hold:

a) for $1<\sigma<1+\mu<2$ the Eigenvalue is greater than +1;

b) for $1<1+\mu<\sigma$, the Eigenvalue is less than +1 for sufficiently low values of ϕ, in particular for

$$0<\phi<\phi^{3s,+1}=\frac{(\sigma-\mu)(1-\sigma+\mu)}{2\mu(1-2\sigma)-\mu^2-\sigma(\sigma-1)}<1 \tag{29}$$

and the Eigenvalue is greater than –1 for sufficiently low values of γ; in particular for:

$$0<\gamma<$$
$$<\gamma^{3s,-1}=\frac{2(\sigma-1)(2\phi+1)(\sigma-\mu+\mu\phi+2\sigma\phi)}{(1-\phi)\left(\left(2\mu(1-2\sigma)-\mu^2-\sigma(\sigma-1)\right)\phi+(\sigma-\mu)(\sigma-1-\mu)\right)} \tag{30}$$

where the inequality $\gamma^{3s,-1}>0$ can be easily verified.

3. 3-regions asymmetric interior fixed points

The industrial activity takes place in all three regions. Two regions exhibit the same share of industrial activity, whereas the other region exhibits a different share:

$$\left(\lambda_1^*,\frac{1-\lambda_1^*}{2},\frac{1-\lambda_1^*}{2}\right)\ \left(\frac{1-\lambda_2^*}{2},\lambda_2^*,\frac{1-\lambda_2^*}{2}\right)\ \left(\frac{1-\lambda_3^*}{2},\frac{1-\lambda_3^*}{2},\lambda_3^*\right)$$

From now on we concentrate our discussion on the first asymmetric fixed point, by symmetry its properties apply also to the other two. The asymmetric fixed point $\left(\lambda_1^*,\frac{1-\lambda_1^*}{2},\frac{1-\lambda_1^*}{2}\right)$ satisfies the following condition:

$$Fpa\left(\lambda_1^*,\mu,\phi,\sigma\right)=0 \qquad \text{with}$$

$$Fpa\left(\lambda_1^*,\mu,\phi,\sigma\right)=\left(\frac{2\left(\phi+\lambda_1^*-\phi\lambda_1^*\right)}{\phi-\lambda_1^*+\phi\lambda_1^*+1}\right)^{\frac{\mu}{\sigma-1}}+ \tag{31}$$
$$+\frac{(1-\phi)\left(2(\mu-\sigma)+4\mu\phi-\sigma\phi\right)\lambda_1^*+3\sigma\phi(\phi+1)}{(1-\phi)(\mu-\sigma+2\mu\phi+4\sigma\phi)\lambda_1^*+(\sigma-\mu)(1+\phi)+2\phi^2(\mu+2\sigma)}$$

It is not possible to derive an explicit expression for the asymmetric fixed point or a manageable expression for the Jacobian evaluated at this fixed point. However, it can be shown analytically that for $\phi=\phi^{CP,+1}$ the asymmetric fixed point coincides with the boundary fixed point (1, 0, 0). In addition, numerical explorations suggest the following properties:

A) for $\phi<\phi^{CP,+1}$ no 3-regions asymmetric fixed point exists;

B) for $\phi^{CP,+1}<\phi<\phi^{3s,+1}$ one 3-regions asymmetric fixed point exists;

C) for $\phi^{3s,+1}<\phi$ one, two or no 3-regions asymmetric fixed point exists.

The following figure 2 depicts $Fpa\left(\lambda_1^*,\mu,\phi,\sigma\right)$ for $\mu=0.45$, $\sigma=2.5$ and for different values of ϕ and illustrates the existence properties (for better visibility $1000\cdot Fpa\left(\lambda_1^*,\mu,\phi,\sigma\right)$ is plotted; note that the 3-regions symmetric fixed point is visible as well).

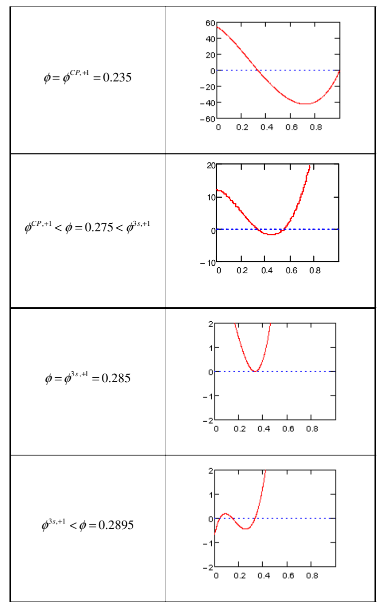

Fig. 2. Existence of 3-regions asymmetric fixed points

4. 2-regions interior fixed points

In the long-run equilibrium, industrial activity is equally shared between two regions, whereas in the third one industrial activity is absent:

$$(0.5, 0.5, 0)\ (0.5, 0, 0.5)\ (0, 0.5, 0.5) \tag{32}$$

The existence of this type of fixed point can easily be confirmed analytically.

The two distinct eigenvalues of the Jacobian evaluated at the fixed point can be determined explicitly and are given by:

$$EV_1^{2reg}=1-\gamma\frac{1-\phi}{1+\phi}\frac{\left[(2-7\mu)\sigma-2\sigma^2+\mu(4-3\mu)\right]\phi-(\sigma-\mu)(3\mu-2\sigma+2)}{3(\sigma-1)(\sigma-\mu+\mu\phi+\sigma\phi)}$$

$$EV_2^{2reg}=1-\gamma\left(1-\left(\frac{2\phi}{1+\phi}\right)^{\frac{\mu+\sigma-1}{\sigma-1}}\frac{\sigma-\mu+2\mu\phi^2+4\sigma\phi^2-\mu\phi+\sigma\phi}{6\sigma}\right)$$

It is possible to show that both eigenvalues are larger than –1 for a sufficiently low γ. However, numerical explorations suggest that for all parameter values at least one of the two eigenvalues is greater than +1. Therefore, it seems that this fixed point is never stable.

5. Summary of local stability properties and basins of attraction for fixed points

Given a sufficiently low value of γ both the Eigenvalues for the 3-regions symmetric equilibrium, for the 2-regions equilibrium and for the boundary equilibrium are greater than -1. For the $+1$ threshold we have the following:

	CP equilibrium	3-regions symmetric equilibrium	3-regions asymmetric equilibrium, 2-regions symmetric equilibrium
$1<\sigma<1+\mu$	EV<+1 always	EV>+1 always	Never stable
$1<1+\mu<\sigma$	EV<+1 for $\phi>\phi^{CP,+1}$	EV<+1 for $\phi<\phi^{3s,+1}$	

Tab. 1. Local stability properties of fixed points

Therefore, for $1<1+\mu<\sigma$ it is possible that both the boundary and the 3-regions symmetric equilibrium are (locally) stable. Figure 3 illustrates this possibility for $\sigma = 2.5$.

In the top left diagram values of ϕ below the dotted line imply stability of the 3-regions symmetric equilibrium and above the solid line imply stability of the boundary equilibrium. Therefore, parameter values in between the two lines represent the parameter set for which both the boundary and the 3-regions symmetric equilibrium are locally stable (provided the value of γ is sufficiently low). The coloured diagrams, drawn for $\gamma=10$ and $\mu=0.45$;[1] and for $\phi=0.25$, $\phi=0.275$ and $\phi=0.3$ respectively, illustrate the basins of attraction. The abscissa represents the initial value for λ_1 and the ordinate that one for λ_2; a yellow tile indicates initial conditions for time paths converging to the 3-regions symmetric equilibrium; the other tiles represent initial conditions for time paths converging to the three boundary equilibria represented by the corners of the triangle. The black points indicate the (numerically determined) 3-regions asymmetric fixed points.

As shown figure 3, these asymmetric fixed points enjoy the following properties: when $\phi=\phi^{CP,+1}$ they appear simultaneously on the three vertexes of the triangle. As trade freeness is increased in the range $\phi^{CP,+1}<\phi<\phi^{3s,+1}$ they travel along the medians towards the centroid of the triangle where, at $\phi=\phi^{3s,+1}$, they merge with the 3-regions symmetric equilibrium. As ϕ is further increased three distinct asymmetric equilibriums reappear doubling in number as they move towards the midpoint of the triangle sides where their number halves again. Finally, a further increase of the trade freeness parameter determines their disappearance.

We conjecture that for $\phi^{CP,+1}<\phi<\phi^{3s,+1}$ the stable manifolds of the 3-regions asymmetric equilibria delimitate the basin of attraction of the 3-regions symmetric equilibrium. Indeed, for $\phi=0.25$ and $\phi=0.275$ (both values lie below $\phi^{3s,+1}\approx 0.285$) the boundary equilibria are (locally) stable as well as the 3-regions symmetric equilibrium; the basin of attraction for the latter (former) shrinks (expands) as ϕ increases. In that parameter range three (locally unstable) 3-regions asymmetric equilibria exist.

For $\phi=0.2895$ and $\phi=0.3$ (both values greater than $\phi^{3s,+1}$) the boundary equilibria are still (locally) stable but the 3-regions symmetric equilibrium is locally unstable; in correspondence of this parameter values, there exist two pairs of (locally unstable) 3-regions asymmetric equilibria and no such an equilibrium, respectively. For this case the figure suggests that the respective basins of attraction of the boundary equilibria are delimitated by segments of the triangle medians as follows:

$$\text{For}\quad 0<\lambda_1<\frac{1}{3}\qquad \lambda_2=\frac{1-\lambda_1}{2},\quad \lambda_3=\lambda_2 \tag{33}$$

$$\text{for}\quad \frac{1}{3}<\lambda_1<\frac{1}{2}\qquad \lambda_2=1-2\lambda_1,\quad \lambda_3=\lambda_1 \tag{34}$$

$$\text{for}\quad \frac{1}{3}<\lambda_1<\frac{1}{2}\qquad \lambda_2=\lambda_1,\quad \lambda_3=1-2\lambda_1 \tag{35}$$

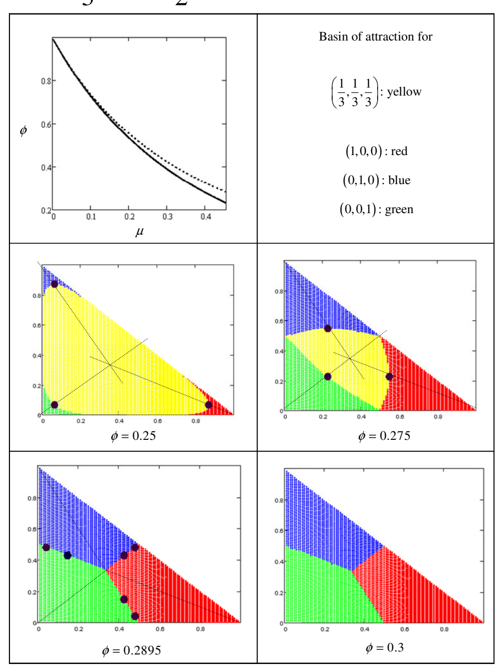

Fig. 3. Fixed points stability properties and basins of attraction

VI. Periodic and Complex Attractors

From equation (28) a limiting value for ϕ can be determined at which the symmetric 3-regions equilibrium loses stability via a flip bifurcation:

[1] This value of μ is close to the value that violates the non-full specialization condition $\mu<\sigma/(3\sigma-2)=5/11$.

$$\phi^{3s,-1} = 1 + \frac{3}{2}\frac{\mu(1-2\sigma)(\gamma-1) - 8\sigma(1-\sigma) - \mu}{\left(4\mu(\gamma-1)+\sigma(\gamma-8)\right)(\sigma-1)+\gamma\mu(\mu+2)} - $$
$$- \frac{\sqrt{4\left(\mu^2+2\gamma\sigma^2\right)(\sigma-1)^2 + 4\gamma\mu^2(\sigma-1) + \gamma^2\mu^2(2\sigma-1)^2}}{\left(4\mu(\gamma-1)+\sigma(\gamma-8)\right)(\sigma-1)+\gamma\mu(\mu+2)} \tag{36}$$

The bifurcation diagram in figure 4, drawn for $\sigma = 8$, $\mu = 0.25$, $\gamma = 5$ (implying $\phi^{Flip}_{sym} \approx 0.1466$), and for $0.1 \le \phi \le 0.15$ illustrates possible dynamic patterns after a flip bifurcation.

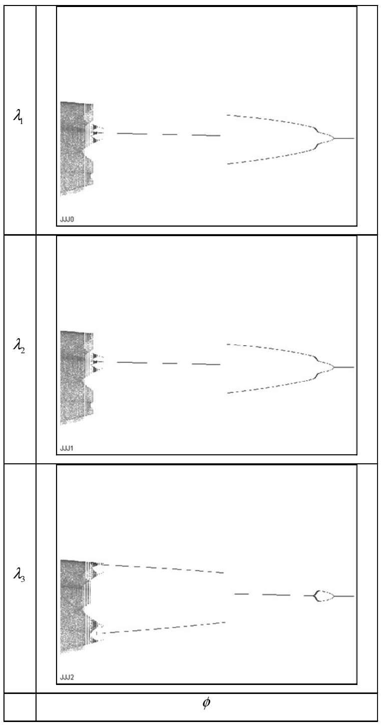

Fig.4. Dynamic behavior after a flip bifurcation

Figure 5 illustrates the dynamic behaviour for particular values of ϕ; red lines (dots) indicate λ_1, blue lines (dots) λ_2, and green lines (dots) λ_3. For $\phi = 0.145$ the time path for all three share exhibits a period-2 cycle; the cycles for λ_2 and λ_3 are identical and exhibit a lower amplitude than the cycle for λ_1.

For $\phi = 0.135$, λ_3 is constant, only λ_1 and λ_2 follows period-2 cycles (identical but with a phase shift).

For lower values of ϕ the dynamic behaviour gets increasingly complex: the bottom panels plot $\lambda_{i,t}$ on the abscissa and $\lambda_{i,t+1}$ on the ordinate.

For $\phi = 0.105$, all three shares settle down on complex attractors, λ_1 and λ_2 on the same two parts attractor; λ_3 on a one part attractor with a lower amplitude.

For $\phi = 0.1$, all three shares settle down on the same one part attractor.

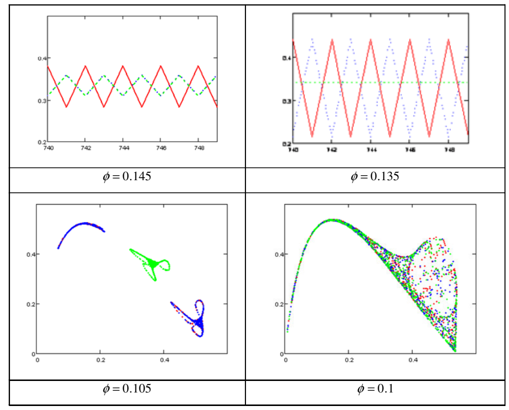

Fig. 5. Time paths and attractors

In figure 6 (which has been plotted for $\mu = 0.35$, $\sigma = 0.8$ and initial condition $\lambda_{1,0} = 0.334$, $\lambda_{2,0} = 0.333$ and $\lambda_{3,0} = 0.333$), we present bifurcation diagrams for $\lambda_{1,t}$, $\lambda_{2,t}$ and $\lambda_{3,t}$ and for $0.35 \le \phi \le 0.55$.

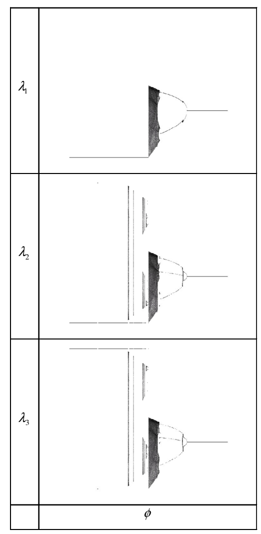

Fig. 6. Bifurcation diagrams with different sets of attractors

Note that the (symmetric) 3-regions dynamics involves regions two and three notwithstanding the initial condition gives a smaller advantage to region 1. Moreover, there appear to be two sets of attractors – one created via the bifurcation of the interior 3-regions fixed point, and another one born via a bifurcation of the 2-regions symmetric fixed point, the latter being visible in the complex 2-regions dynamics. Finally, for a low ϕ, industrial activity is agglomerated only in one region.

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